The XXZ Heisenberg model on random surfaces.

J. Ambjørn [a,c], and **A. Sedrakyan** [a,b]

[a] The Niels Bohr Institute, Copenhagen University
Blegdamsvej 17, DK-2100 Copenhagen, Denmark.
email: ambjorn@nbi.dk

[b] Yerevan Physics Institute
Br. Alikhanyan str 2, Yerevan-36, Armenia.
email: sedrak@nbi.dk

[c] Institute for Mathematics, Astrophysics and Particle Physics (IMAPP)
Radbaud University Nijmegen
Heyendaalseweg 135, 6525 AJ, Nijmegen, The Netherlands

Abstract

We consider integrable models, or in general any model defined by an R-matrix, on random surfaces, which are discretized using random Manhattan lattices. The set of random Manhattan lattices is defined as the set dual to the lattice random surfaces embedded on a regular d-dimensional lattice. They can also be associated with the random graphs of multiparticle scattering nodes. As an example we formulate a random matrix model where the partition function reproduces the annealed average of the XXZ Heisenberg model over all random Manhattan lattices. A technique is presented which reduces the random matrix integration in partition function to an integration over their eigenvalues.

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1 Introduction

One of the major goals of non-critical string theory was to describe the non-perturbative physics of non-Abelian gauge fields in two, three and four dimensions. The asymptotic freedom of these theories allowed us to understand the scattering observed at high energies [1, 2], but it also made the long distance, low energy sector of the theories non-perturbative and indicated a non-trivial structure of the vacuum [3, 4]. It became necessary to develop non-perturbative tools which would allow us to study phenomena associated with e.g. confinement. One possibility which attracted a lot of attention was the attempt by Polyakov to reformulate the non-Abelian gauge theories as a string theory. This line of research led Polyakov to his seminal work on non-critical string theory [5]. He showed that the presence of the conformal anomaly forces us to include the conformal factor in the string path integral, and that the action associated with the conformal factor is the Liouville action. In his approach the study of non-critical string theory becomes equivalent to the study of two-dimensional quantum gravity (governed by the Liouville theory) coupled to certain conformal matter fields.

Attempts to understand and define rigorously the quantum Liouville theory triggered the lattice formulation, presenting the two-dimensional random surfaces appearing in the string path integral as a sum over triangulated piecewise linear surfaces [6, 7, 9]. This sum over "random triangulations" (or "dynamical triangulations" (DT)) could be represented by matrix integrals and in this way certain matrix integrals became almost synonymous to non-critical string theory. Somewhat surprisingly many of the lattice models were exactly solvable and at the same time it was possible to solve the continuum quantum Liouville model via conformal bootstrap, and whenever results of the two non-pertubative methods could be compared, agreement was found. However, the solutions only made sense for matter fields with a central charge $c \leq 1$ [12]. Thus, in a certain way the approach of Polyakov came to a stalemate with regards to the question of using non-critical string theory to understand the non-perturbative aspects of three and four-dimensional non-Abelian gauge theories, which seemingly require $c > 1$. It is thus of great interest to try to study new classes of random surface models which might allow us to penetrate the $c = 1$ barrier. This is one of the main motivations of this paper. We propose to consider a new class of random lattices, the so-called random Manhattan lattices. One is led to such lattices by studying the random surface representation of the 3d Ising model on a regular 3d lattice [13, 14], and via the study of the Chalker-Coddington network model [16]. The study of the latter model led to the idea that an R-matrix could be associated to a random Manhattan lattice, and we will consider how to couple in general a matter system defined by an R-matrix to a random lattice. By summing over the random lattices (i.e. taking the annealed average) we thus introduce a coupling
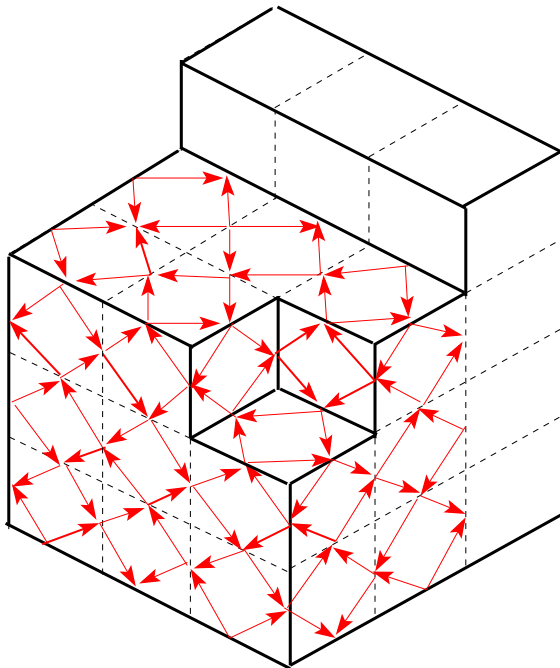
Figure 1: A random surface on the 3d cubic lattice and the construction of the dual lattice surface, as described in the text.

between the integrable model and two-dimensional quantum gravity.

More precisely we start with an integrable model on a 2d square lattice, assuming we know the R-matrix. We then show that the same R-matrix can be used on a so-called Random Manhattan Lattice (RML) (see Fig. 1), which is a lattice where the links have fixed arrows which indicate the allowed fermion hopping. No hopping is allowed in directions opposite to arrows. The summation over the RMLs can be performed by a certain matrix integral related to the R-matrix. This matrix integral is somewhat different from the the conventional matrix integrals used to describe conformal field theories with $c < 1$ coupled to 2d quantum gravity, and thus there is hope than one can penetrate to $c = 1$ barrier. Below we describe the construction in detail.

2 The model

As mentioned one arrives in a natural way to a RML from the study of the 3d Ising model on a regular cubic lattice. The high temperature expansion of the Ising model can be expressed as a sum over random lattice surfaces of the kind shown in Fig. 1, and on these two-dimensional lattice surfaces one constructs a kind of dual lattice by the following procedure: The lattice surface consists of

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plaquettes. Consider the mid-points of the links on the plaquettes as sites of the dual lattice, and consider arrows on the links as shown in Fig. 2.
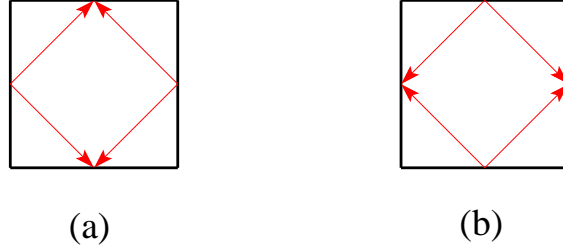

<table>
<tr><td>(a)</td><td>(b)</td></tr>
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Figure 2: Assignment of arrows to dual lattice

The allover orientation of arrows on the plaquettes should be such that the flow to neighbouring plaquettes is continuous as illustrated in Fig. 3. This type of dual lattice with arrows will be a finite Manhattan lattice corresponding to a particular plaquette lattice surface on the regular three-dimensional lattice. There is a one to one correspondence between the plaquette surfaces on the regular lattice and the finite Manhattan lattices described above.

A second way of obtaining a RML is by starting from oriented double line graphs, like the ones introduced by 't Hooft, and then modify the double line propagator like shown in Fig. 4.

We will now attach an R-matrix of an integrable model to the squares of the RML with the index assignment shown in Fig. 5. Two neighbouring squares will share one of indices, and the same is thus the case for the corresponding R-matrices, and a summation over values of the indices are understood, resulting in a matrix-like multiplication of R-matrices. To a RML Ω we now associate the partition function

$$Z(\Omega) = \prod_{R \in \Omega} \check{R}, \tag{1}$$

where the summation over indices is dictated by the lattice. Our final partition function is defined by summing over all possible (connected) lattices Ω:

$$Z = \sum_{\Omega} Z(\Omega) \, \mathrm{e}^{-\mu|\Omega|} \tag{2}$$

where μ is a "cosmological" constant which monitors the typical size $|\Omega|$ of the lattice Ω. As long as we restrict the topology of lattices Ω entering in the sum (2), there will exist a critical μ_c such that the sum in (2) is convergent for $\mu > \mu_c$ and divergent for $\mu < \mu_c$. We will be interested in a limit where the average value of Ω becomes infinite, and this limit is obtained by approaching μ_c from above. The summation over the elements in Ω, i.e. the summation over a certain set of random
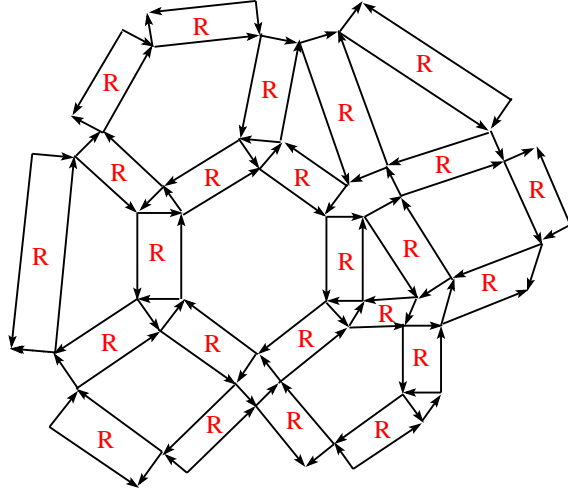
Figure 3: Random Manhattan lattice
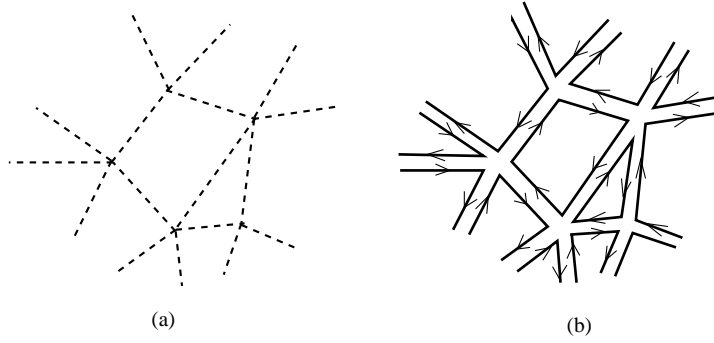

(a)   (b)

Figure 4: Manhattan lattices and double line graphs

2d lattices, is a regularized version of the sum over 2d geometries precisely in the same way as in ordinary DT. It is known the sum over random polygons (triangles, squares, pentagons etc) with positive weights under quite general conditions leads to the correct continuum limit, i.e. the functional integral over 2d geometries, when the link length goes to zero. Thus it is natural to assume that the sum over RML will also represent in correct way the sum over 2d geometries when the link length goes to zero. Under this assumption we have coupled a given two-dimensional model, integrable on a regular lattice, to two-dimensional quantum gravity. In the next Section we will, in order to make the discussion more explicit, consider the *XXZ* Heisenberg model, where the R-matrix is known.
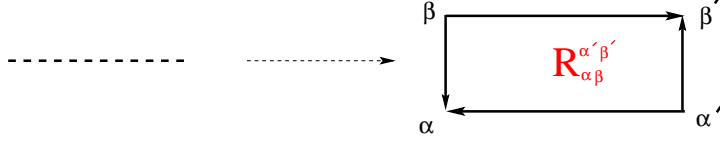
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Figure 5: Index assignment of the R-matrix

3   The matrix model

In order to represent the *XXZ* model as a matrix model which at the same time will offer us a topological expansion of the surfaces spanned by the oriented ribbon graphs considered above, we consider the set of $2N \times 2N$ normal matrices. We label the entries of the matrices as $M_{\alpha\beta,ij}$, where α, β takes values 0,1 and i, j takes values $1, \ldots, N$. The α, β indices refer to the *XXZ* model, while the i, j indices will be used to monitor the topological expansion. A normal matrix is a matrix with complex entries which can be diagonalized by a unitary transformation, i.e. for a given normal matrix M there exists a decomposition

$$M = U M^{(d)} U^{\dagger} \tag{3}$$

where U is a unitary $2N \times 2N$ matrix and $M^{(d)}$ a diagonal matrix with eigenvalues $m^{(d)}_{\alpha,ii}$ which are complex numbers.

Consider now the action

$$S(M) = M^{*}_{\alpha\beta,ij} \check{R}^{\alpha'\beta'}_{\alpha\beta} M_{\beta'\alpha',ij} - \sum_{n=3}^{\infty} \frac{d_n}{n} \, \mathrm{tr} \left(M^n + (M^{\dagger})^n \right). \tag{4}$$

We denote the sum over traces of M and $M^{\dagger}$ as the potential. The matrix partition function is defined by

$$Z = \int \mathrm{d}M \, \mathrm{e}^{-NS(M)}. \tag{5}$$

When one expands the exponential of the potential terms and carries out the remaining Gaussian integral one will generate all graphs of the kind discussed above, with the R-matrices attached to the graphs as described. The only difference is that the graphs will be ordered topologically such that the surfaces associated with the ribbon graphs appear with a weight N^{χ}, where χ is the Euler characteristics of the surface. If we are only interested in connected surfaces we should use as the partition function

$$F = \log Z. \tag{6}$$

In particular the so-called large N limit, which selects connected surfaces with maximal χ, will sum over to the planar (connected) surfaces generated by F, since these are the connected surfaces with the largest χ.

Explicitly, in the case of the XXZ Heisenberg model the R matrix is given by:

$$
\begin{aligned}
\check{R}^{\alpha'\beta'}_{\alpha\beta} &= \frac{a+c}{2} 1^{\alpha'}_{\alpha} \otimes 1^{\beta'}_{\beta} + \frac{a-c}{2} (\sigma_3)^{\alpha'}_{\alpha} \otimes (\sigma_3)^{\beta'}_{\beta} \\
&\quad \frac{b}{2} \left((\sigma_1)^{\alpha'}_{\alpha} \otimes (\sigma_1)^{\beta'}_{\beta} + (\sigma_2)^{\alpha'}_{\alpha} \otimes (\sigma_2)^{\beta'}_{\beta} \right).
\end{aligned}
\tag{7}
$$

As an abbreviation we will write

$$
\check{R}^{\alpha'\beta'}_{\alpha\beta} = \sigma_a \otimes \sigma_a \tilde{I}_a
\tag{8}
$$

where a summation over index a is understood, $\sigma_0 = 1$, the identity matrix, and σ_a, $a = 1, 2, 3$ are the Pauli matrices.

Our aim is to decompose the integration over the matrix entries of M into their radial part M_d and the angular U-parameters. This decomposition is standard, the Jacobian is the so-called Vandermonde determinant (also in the case of normal matrices, [17]) When we make that decomposition the potential will only depend on the eigenvalues $m^{(d)}_{\alpha,ii}$ and for the measure we have:

$$
\mathrm{d}M = \mathrm{d}U \prod_{\alpha,i} \mathrm{d}m^{(d)}_{\alpha,ii} \, \mathrm{d}m^{(d)*}_{\alpha,ii} \prod_{\alpha,i\neq\beta,j} \left| (m^{(d)}_{\alpha,ii} - m^{(d)}_{\beta,jj}) \right|^2.
\tag{9}
$$

However, the problem compared to a standard matrix integral is that the matrices U, introduced by the transformation (3), will appear quartic in the action (4). Thus the U-integration does not reduce to an independent factor, decoupled from the rest. Neither is it of the Itzykson-Zuber-Charish-Chandra type.

In order to perform the integral we pass from the transformation (3) which is given in the fundamental representation, to a form where we use the adjoint representation. Let us choose a basis t^A for Lie algebra of the unitary group $U(2N)$ in the fundamental representation. The normal matrix M can also be expended in this basis:

$$
M = C_A t^A, \quad \mathrm{tr}\, t^A t^B = \delta^{AB},
\tag{10}
$$

where the last condition just is a convenient normalization. For a given U belonging to the fundamental representation of $U(2N)$ the corresponding matrix in the adjoint representation, $\Lambda(U)$, and the transformation (3) are given by

$$
\Lambda(U)_{AB} = \mathrm{tr}\, t^A U t^B U^\dagger, \quad C_A = \Lambda_{AB} C^{(d)}_B,
\tag{11}
$$

where $C_B^{(d)}$ denotes the coordinates of $M^{(d)}$ in the decomposition (10). The transformation (3) is now linear in the adjoint matrix Λ and the action (4) will be quadratic in Λ. However, we pay of course a price, namely that the entries of the $(2N)^2 \times (2N)^2$ unitary matrix Λ satisfy more complicated constraints than those satisfied by the entries of the $2N \times 2N$ unitary matrix U. We will deal with the this problem below. First we express the action (4) in terms of the eigenvalues $m_{\alpha,ii}^{(d)}$ and Λ.
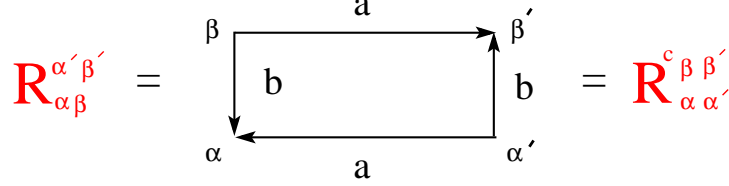

Figure 6: Index assignment of the R-matrix

For this purpose it is convenient to pass from the representation (7) of the R-matrix to the cross channel (see Fig. 6), which is achieved by:

$$\left(\check{R}^c\right)^{\beta\beta'}_{\alpha\alpha'} = \check{R}^{\alpha'\beta'}_{\alpha\beta}, \tag{12}$$

which amounts to making a particle-hole transformation $0 \leftrightarrow 1$ for the indices α' and β. After some algebra one obtains:

$$\begin{aligned}\left(\check{R}^c\right)^{\beta\beta'}_{\alpha\alpha'} &= \frac{b+c}{2}\sigma_1 \otimes \sigma_1 + \frac{b-c}{2}\sigma_2 \otimes \sigma_2 \\ &\quad \frac{a}{2}\left[1 \otimes 1 + \sigma_3 \otimes \sigma_3\right].\end{aligned} \tag{13}$$

We can now write

$$\left(\check{R}^c\right)^{\beta\beta'}_{\alpha\alpha'} = \sigma_a \otimes \sigma_a \, I^a, \tag{14}$$

where the summation is over a, $a = 0, 1, 2, 3$, $\sigma_0 \equiv 1$, the identity 2×2 matrix. One has:

$$I_a = \left(\frac{a}{2}, \frac{b+c}{2}, \frac{b-c}{2}, \frac{a}{2}\right) \tag{15}$$

Using the cross channel R-matrix (14) the action (4) becomes:

$$S = M^*_{\alpha\beta,ij}\left(\check{R}^c\right)^{\beta\beta'}_{\alpha\alpha'} \delta^{i'}_i \delta^{j'}_j M_{\beta'\alpha',i'j'} - \mathrm{Re}V(M). \tag{16}$$

Let τ^μ, $\mu = 1, \ldots, N^2$ denote generators of the Lie algebra of $U(N)$, appropriately normalized such that

$$\delta^{i'}_i \delta^{j'}_j = \tau^\mu_{ij}\tau^\mu_{i'j'}. \tag{17}$$

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If we insert (17) into the action (16) we obtain

$$S = \text{tr} \left(M^\dagger \sigma^a \tau^\mu \right) I^a \text{tr} \left(\sigma^a \tau^\mu M \right) \tag{18}$$

where we can view $\sigma^a \tau^\mu$, $a = 0, 1, 2, 3$ and $\mu = 1, \ldots, N^2$ as the $(2N)^2$ generators t^A of the Lie algebra of $U(2N)$. Formulas (10) and (11) with the generators $\sigma^a \tau^\mu$ read

$$m_{a,\mu}^{(d)} = \frac{1}{2} \text{tr} \left(M^{(d)} \sigma^a \tau^\mu \right), \tag{19}$$

$$\Lambda^{a\mu,a'\mu'} = \frac{1}{2} \text{tr} \left(\sigma^a \tau^\mu U \sigma^{a'} \tau^{\mu'} U^\dagger \right). \tag{20}$$

We now want to use (3) and (19) and (20) to express the matrix M in the action (18) in terms of $m_{a,\mu}^{(d)}$ and $\Lambda^{a\mu,a'\mu'}$, and we obtain

$$S = m_{a'\mu'}^{(d)*} \Lambda^{a\mu,a'\mu'} I^a \Lambda^{a\mu,a''\mu''} m_{a''\mu''}^{(d)} - V(m_{a\mu}^{(d)}) \tag{21}$$

It is convenient to choose following basis for generators t_A, $A \equiv (a, ij) = 1 \cdots 4N^2$ of $U(2N)$. For Cartan sub-algebra we take $t_{0,11} = 1 \otimes 1$ for common phase factor, $t_{3,ii} = \sigma_3 \otimes \tau^{ii}$, $i = 1 \cdots N$ and $t_{0,ii} = \frac{1}{2}(1+\sigma_3) \otimes (\tau^{i,i} - \tau^{i-1,i-1})$, $i = 2 \cdots N$ for other $2N - 1$ traceless generators, where

$$\left(\tau^{ij} \right)_{kl} = \delta_{ik} \delta_{jl}. \tag{22}$$

For the remaining, non-diagonal generators we take $\sigma_a \tau^{ij}$, $a = 1, 2$, $i, j = 1 \cdots N$ and $\sigma_a \tau^{ij}$, $a = 0, 1, 2, 3$, $i \neq j = 1 \cdots N$. Then, for this choice of generators and from $m_{a,ij}^{(d)} = \frac{1}{2} \text{tr} \left(M^{(d)} t_{a,ij} \right)$ we have

$$m_{0,11}^{(d)} = \text{tr}\, M = \sum (m_{\alpha=1,ii}^{(d)} + m_{\alpha=2,ii}^{(d)}) \tag{23}$$

$$m_{3,ii}^{(d)} = m_{\alpha=1,ii}^{(d)} - m_{\alpha=2,ii}^{(d)}, \tag{24}$$

$$m_{0,ii}^{(d)} = m_{\alpha=1,ii}^{(d)} - m_{\alpha=1,i-1i-1}^{(d)} \tag{25}$$

$$m_{b=1,ii}^{(d)} = m_{b=2,ii}^{(d)} = 0 \tag{26}$$

with the rest of elements $m_{a,ij}^{(d)} = 0$, $a = 0, 1, 2, 3$, $i \neq j = 1, \cdots N$.

3.1   The space of integration

We now change the integration over the unitary matrices $U(2N)$ in formula (9), which are in the fundamental representation, to the unitary matrices $\Lambda(2N)$ in the adjoined representation. Rather than using the Haar measure expressed in terms of the U-matrices we should express the Haar measure in terms of the Λ-matrices.

Since normal matrices M can be regarded as elements in the algebra $u(2N)$, the action of Λ, defined by the formula (11) on its diagonalized form (23), will form an orbit in the algebra with the basis consisting of all diagonal matrices. Diagonalized elements of M are invariant under the action of the maximal abelian (Cartan) subgroup $\otimes U(1)^{2N}$ of $U(2N)$. Therefore the orbits are isomorphic to the factor space $\frac{U(2N)}{U(1)\otimes\cdots\otimes U(1)}$.

Moreover this factor space is isomorphic to a so-called flag-manifold, defined as follows (see [19, 20] and references there): A single flag is a sequence of nested complex subspaces in a complex vector space C_n

$$\{\varnothing\} = C_0 \subset C_{a_1} \subset \cdots \subset C_{a_k} \subset C_n = C^n \tag{27}$$

with complex dimensions $dim_C C_i = i$. For a fixed set of integers $(a_1, a_2 \cdots a_k, n)$ the collection of all flags forms a manifold, which called a flag manifold $F(n_1, n_2, \cdots n_k)$, where $n_i = a_i - a_{i-1}$. The manifold $F(1, 1, \cdots 1)$ is called a full flag manifold, others are partial flag manifolds. The full flag manifold $F(1, 1, \cdots 1)$ is isomorphic to the orbits of the action of the adjoined representation of $U(2N)$ on its algebra

$$F(1, 1, \cdots 1) = \frac{U(2N)}{\otimes U(1)^{2N}} \tag{28}$$

The set of C_i hyperplanes in C_{i+1} is isomorphic to the set of complex lines in C_{i+1}. In differential geometry this set is denoted by $\mathbf{CP}^i$ (and also as Grassmanians $\mathbf{Gr}(1, i)$) and is called a complex projective space. Hence, the complex projective space is a factor space

$$\mathbf{CP}^i = \frac{U(i+1)}{U(i) \otimes U(1)} = \frac{S^{2i+1}}{U(1)} \tag{29}$$

where S^{2i+1} is a real $2i+1$ dimensional sphere.

According to description presented above the orbit of the action of the adjoined representation of $U(2N)$ on the set of normal matrices M is a sequence of fiber bundles and locally, on suitable open sets, the elements of the flag manifold can be represented as a direct product of projective spaces (the fibers)

$$\frac{U(2N)}{\otimes U(1)^{2N}} \simeq \mathbf{CP}^{2N-1} \times \mathbf{CP}^{2N-2} \times \cdots \mathbf{CP}^1 \tag{30}$$

In simple words we have the following representation of the orbit: any diagonalized normal matrix in the adjoined representation has a following form

$$
M_{a\mu}^{(d)} = \Big(\underbrace{m_{3,NN}^{(d)}, 0, \cdots 0}_{4N-1}, \underbrace{m_{0,NN}^{(d)}, 0, \cdots 0}_{4N-3}, \cdots \underbrace{m_{3,kk}^{(d)}, 0, \cdots 0}_{4k-1}, \underbrace{m_{0,kk}^{(d)}, 0, \cdots 0}_{4k-3},
$$
$$
\cdots \underbrace{m_{3,11}^{(d)}, 0, 0, m_{0,11}^{(d)}}_{3} \Big) \tag{31}
$$

The action of the adjoined representation Λ on this M transforms it into the elements of $\frac{U(2N)}{\otimes U(1)^{2N}}$ presented in (30) where $\mathbf{CP}^{2k-1}$ represents image of the part $\underbrace{m_{3,kk}^{(d)}, 0, \cdots 0}_{4k-1}$.

This implies that the measure of our integral over normal matrices M can be decomposed into the product of measures of the base space (the diagonal matrices) and the flag manifold (the fiber)

$$
\begin{aligned}
\mathcal{D}\Lambda &= \prod_{i=1}^{N} dm_{0,ii}^{(d)} dm_{3,ii}^{(d)} \prod_{k=1}^{2N-1} \mathcal{D}[\mathbf{CP}^k] \\
&= \prod_{i=1}^{N} dm_{0,ii}^{(d)} dm_{3,ii}^{(d)} \prod_{k=1}^{N} \mathcal{D}\left[\frac{S^{4k-1}}{S^1}\right] \mathcal{D}\left[\frac{S^{4k-3}}{S^1}\right]
\end{aligned} \tag{32}
$$

However, since the diagonal matrix elements $m_{0,ii}^{(d)}$ and $m_{3,ii}^{(d)}$ are complex, our action is invariant over $\otimes U(1)^{2N}$ (one $U(1)$ per marked segment in (31) and we can extend the integration measure from (32) to

$$
\mathcal{D}\Lambda = \prod_{i=1}^{N} dm_{0,ii}^{(d)} dm_{3,ii}^{(d)} \prod_{k=1}^{N} \mathcal{D}\left[S^{4k-1}\right] \mathcal{D}\left[S^{4k-3}\right] \tag{33}
$$

In other words, we suggest that the action of Λ on the segments $\underbrace{m_{3,kk}^{(d)}, 0, \cdots 0}_{4k-1}$ and $\underbrace{m_{0,kk}^{(d)}, 0, \cdots 0}_{4k-3}$ in (31) form vectors $m_{3,kk}^{(d)} z_{3,k}^r$, $(r = 1 \cdots 4k-1)$ and $m_{0,kk}^{(d)} z_{0,k}^s$, $(s = 1 \cdots 4k-3)$, respectively, where the real coordinates $z_{3,k}^r$ and $z_{0,k}^r$ belong to the unite spheres S^{4k-1} and S^{4k-3}.

In order to write the measure of integration over the spheres S^{4k-1} and S^{4k-3} we embed them into the Euclidean spaces R^{4k} and R^{4k-2}, respectively, and define

$$
\begin{aligned}
\mathcal{D}\left[S^{4k-1}\right] &= \delta\left(\sum_{s=1}^{4k}[z_{3,k}^s]^2 - 1\right) \prod_{s=1}^{4k} dz_{3,k}^s = \int d\lambda_{3,k} \prod_{s=1}^{4k-1} dz_{3,k}^s e^{-\lambda_{3,k}^2(1-\sum_{s=1}^{4k}[z_{3,k}^s]^2)}, \\
\mathcal{D}\left[S^{4k-3}\right] &= \delta\left(\sum_{s=1}^{4k-2}[z_{0,k}^s]^2 - 1\right) \prod_{s=1}^{4k-2} dz_{0,k}^s = \int d\lambda_{0,k} \prod_{s=1}^{4k-2} dz_{0,k}^s e^{-\lambda_{0,k}^2(1-\sum_{s=1}^{4k-3}[z_{0,k}^s]^2)},
\end{aligned} \tag{34}
$$

where we have introduced Gaussian integrations over the real parameters $\lambda_{a,k}$, $a = 0, 3$. These integrations reproduce the factors $\frac{1}{2\sqrt{1-\sum_{s=1}^{4k-3}[z_{a,k}^s]^2}}$ which arise from

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the δ-functions in (34) by integrations over the coordinates $z_{3,k}^{4k}$ and $z_{0,k}^{4k-2}$. We have omitted coefficients $\sqrt{\pi}/2$ in front of integrals on the right hand side of the expressions (34) since they unimportant for the partition function.

With this definition of the measure the partition function (5) can be written as

$$
\int dM e^{-NS(M)} = \int \prod_{k,a=0,3} dm_{a,kk}^{(d)} d\lambda_{a,k} \prod_{s=1}^{4k-1} dz_{3,k}^s \prod_{s=1}^{4k-3} dz_{0,k}^s W(m_{\alpha,kk}^{(d)}) e^{-S(m_{a,kk}^{(d)},\lambda_{a,k},z_{a,k})},
$$
(35)

where $W(m_{\alpha,kk}^{(d)}) = \prod_{\alpha,i\neq\beta,j} \left|(m_{\alpha,ii}^{(d)} - m_{\beta,jj}^{(d)})\right|^2$ is Vandermonde determinant and

$$
S(m_{a,kk}^{(d)},\lambda_{a,k},z_{a,k}) = \sum_{a=0,3,k=1}^{N} \left[|m_{a,kk}^{(d)*}|^2 \sum_{b=1}^{s(a,k)} |z_{a,k}^b|^2 I^b + \lambda_{a,k}^2 \left(1 - \sum_{s=1}^{s(a,k)} [z_{a,k}^s]^2\right) - V(m_{a,kk}^{(d)}) \right].
$$
(36)

Here $s(3,k) = 4k - 1$ and $s(0,k) = 4k - 3$, while according to (15)

$$
I = \frac{1}{2}\Big(\underbrace{a, b-c, b+c, a\cdots b+c}_{4N-1}, \underbrace{a, a, b-c, b+c, a\cdots a}_{4N-3}, \cdots \underbrace{a, b-a, b+a}_{3}, \underbrace{a}_{1}\Big).
$$
(37)

The length of I precisely is $4N^2$. The 4 elements $a/2, (b-c)/2, (b+c)/2, a/2$ of I^b are placed as in a specific sequence in I, as shown in eq. (37). However, the partition function is independent of this choice (which is just our choice arbitrary choice) after integration.

As one can see we have in the partition function (36) simple Gaussian integrals over $z_{a,k}^s$. These can be evaluated and we are left with integrals over $m_{a,kk}^{(d)}$ and $\lambda_{a,k}$ only. It is convenient to rescale the Lagrange multipliers and introduce $\tilde{\lambda}_{a,k} = |m_{b,ii}^{(d)}|^{-1}\lambda_{a,k}$. Then, after performing the Gaussian integrals, we obtain

$$
Z = \int \prod_{a=0,3;k=1}^{N} dm_{a,kk}^{(d)} W(m_{a,kk}^{(d)}) \prod_{k=1}^{N} \frac{1}{|m_{3,kk}^{(d)}|^{4k-2}|m_{0,kk}^{(d)}|^{4k-4}} \int \prod_{a=0,3;k=1}^{N} d\tilde{\lambda}_{a,k}
$$
$$
\cdot \prod_{a=0,3;k=1}^{N} e^{-\text{Re}V(m_{a,kk}^{(d)})-|m_{a,kk}^{(d)}|^2\tilde{\lambda}_{a,k}^2} \prod_{a=0,3;k=1}^{N} Z_{a,k}(\tilde{\lambda}_{a,k}).
$$
(38)

where

$$
Z_{3,k}(\tilde{\lambda}_{3,k}) = \frac{1}{(a - \tilde{\lambda}_{3,k}^2)^{k-\frac{1}{2}}} \frac{1}{(b - c - \tilde{\lambda}_{3,k}^2)^{\frac{k}{2}}} \frac{1}{(b + c - \tilde{\lambda}_{3,k}^2)^{\frac{k}{2}}}
$$
$$
Z_{0,k}(\tilde{\lambda}_{0,k}) = \frac{1}{(a - \tilde{\lambda}_{0,k}^2)^{k-\frac{1}{2}}} \frac{1}{(b - c - \tilde{\lambda}_{0,k}^2)^{\frac{k-1}{2}}} \frac{1}{(b + c - \tilde{\lambda}_{0,k}^2)^{\frac{k-1}{2}}}
$$
(39)

Let us demonstrate that the Gaussian integration over the adjoint representation matrice Λ in (38) correctly reproduces the partition function (5) when interaction is absent, i.e. $V(M) = 0$. In this case the integral over normal matrices M in the fundamental representation of $U(2N)$ can easily be performed directly and the result is:

$$\frac{1}{\sqrt{\prod_{b=1}^{4N^2} I_b}} = \left[a^2(b^2 - c^2)\right]^{-N^2/2}.$$

Let us first consider the simple case $N = 1$. In the general setup this corresponds to having the two shortest, length 3 and 1 segments in the sequence (31). The Vandermonde determinant cancels the m's in the denominator in the expression (38) of the partition function and integration over the m's leads to

$$
\begin{aligned}
Z &= \int \frac{d\tilde{\lambda}_{3,1}}{\tilde{\lambda}_{3,1}} \frac{d\tilde{\lambda}_{0,1}}{\tilde{\lambda}_{0,1}} \frac{1}{(a - \tilde{\lambda}_{3,1}^2)^{1/2}} \frac{1}{(b - c - \tilde{\lambda}_{3,1}^2)^{1/2}} \frac{1}{(b + c - \tilde{\lambda}_{3,1}^2)^{1/2}} \frac{1}{(a - \tilde{\lambda}_{0,1}^2)^{1/2}} \\
&= \frac{1}{[a^2(b^2 - c^2)]^{1/2}}
\end{aligned}
\tag{40}
$$

provided we place the λ-poles at zero and the λ-branch cuts at different sides of the real λ-axis.

Now consider a general N. For a generic i segment in (31) we first represent the multipliers $|m_{1,ii}^{(d)} - m_{1,jj}^{(d)}|$ and $|m_{1,ii}^{(d)} - m_{2,jj}^{(d)}|$ in the Vandermonde determinant as a sum over $m_{a,ii}^{(d)}$, $a = 0, 3$:

$$
|m_{1,ii}^{(d)} - m_{1,jj}^{(d)}| = |m_{0,ii}^{(d)} + \sum_{r=j+1}^{i-1} m_{0,rr}^{(d)}|
$$

$$
|m_{\alpha=1,ii}^{(d)} - m_{\alpha=2,jj}^{(d)}| = |m_{3,ii}^{(d)} + \sum_{r=j+1}^{i-1} (m_{\alpha=2,rr}^{(d)} - m_{\alpha=2,r-1r-1}^{(d)})|.
\tag{41}
$$

When using this decomposition in the Vandermonde determinant product, we observe that only the contribution of the selected terms $|m_{0,ii}^{(d)}|$ and $|m_{3,jj}^{(d)}|$ in (41) will cancel all $m^{(d)}$'s in the denominator of (38) and thus lead to a nonzero contribution. By Cauchy integration as above we obtain

$$
\begin{aligned}
Z &= \int \prod_{a=0,3;i=1}^{N} \frac{d\tilde{\lambda}_{3,i}}{\tilde{\lambda}_{3,i}} \frac{d\tilde{\lambda}_{0,i}}{\tilde{\lambda}_{0,i}} Z_{a,i}(\tilde{\lambda}_{a,i}) \\
&= \frac{1}{[a^2(b^2 - c^2)]^{N^2/2}}.
\end{aligned}
\tag{42}
$$

13

Other terms in the decomposition (41) will result in $\tilde{\lambda}_{a,i}$, $a = 0, 3$ appearing in the denominator of the integral (42) with other powers than one, and the integration will give zero for these terms.

4   Conclusions

We have defined a matrix model which reproduces the partition function of an integrable model (the XXZ model) on random surfaces. The random surfaces under consideration appear as random Manhattan lattices, which are dual to random surfaces embedded in a d dimensional regular Euclidean lattice. This formulation allows us to consider a new type of non-critical strings with $c > 1$.

We have shown that the matrix integral can be reduced to integrals over the eigenvalues of matrices. The important ingredient in the integration over angular parameters of the matrices (which are usually defined by the Itzykson-Zuber integral) is the reduction of the problem to an integration over unitary matrices in the adjoined representation. In principle one can now try to apply standard large N saddle point methods for solving the resulting integrals over eigenvalues.

Thoughout in this paper we have considered the Heisenberg XXZ model on random surfaces, but the approach can be applied to other integrable models.

5   Acknowledgement

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